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Products of integral-type operators and composition operators between Bloch-type spaces

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ABSTRACT

A complete picture on the boundedness and compactness of the products of integral-type operators and composition operators between Bloch-type spaces of holomorphic functions on the unit disk is given in this paper.

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1. Introduction

Let \mathbb{D} be the open unit disk in the complex plane \mathbb{C} and $H(\mathbb{D})$ the class of all analytic functions on \mathbb{D} . An analytic function f on \mathbb{D} is said to belong to the Bloch-type space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$ ($\alpha > 0$), if

$$B_\alpha(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The expression $B_\alpha(f)$ defines a seminorm while the natural norm is given by $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + B_\alpha(f)$. It makes \mathcal{B}^α into a Banach space. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well-known Bloch space (see, for example, [2,7,23,34,35]). Let \mathcal{B}_0^α denote the subspace of \mathcal{B}^α consisting of those $f \in \mathcal{B}^\alpha$ for which

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |f'(z)| = 0.$$

This space is called the little Bloch-type space.

Let $L : X \rightarrow Y$ be a linear operator, where X and Y are Banach spaces. The operator L is said to be compact if for every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in X , the sequence $(L(x_n))_{n \in \mathbb{N}}$ has a convergent subsequence. The operator L is said to be weakly compact if for every bounded sequence $(x_n)_{n \in \mathbb{N}}$ in X , $(L(x_n))_{n \in \mathbb{N}}$ has a weakly convergent subsequence, i.e., there is a subsequence $(x_{n_m})_{m \in \mathbb{N}}$ such that for every $\Lambda \in Y^*$, $\Lambda(L(x_{n_m}))_{m \in \mathbb{N}}$ converges. A useful characterization for an operator to be weakly compact is the following Gantmacher's theorem: L is weakly compact if and only if $L^{**}(X^{**}) \subset Y$, where L^{**} is the second adjoint of L and Y is identified with its image under the natural embedding into its second dual Y^{**} (see [9]).

Let φ be an analytic self-map of \mathbb{D} . Associated with φ is the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces (see, for example, [8]).

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Suppose that $g: \mathbb{D} \rightarrow \mathbb{C}^1$ is a holomorphic map and $f \in H(\mathbb{D})$. The integral-type operator J_g is defined by

$$J_g f(z) = \int_0^z f(\xi) g'(\xi) d\xi, \quad z \in \mathbb{D}.$$

Another integral-type operator I_g is defined by

$$I_g f(z) = \int_0^z f'(\xi) g(\xi) d\xi, \quad z \in \mathbb{D}.$$

The importance of the operators J_g and I_g comes from the fact that

$$J_g f + I_g f = M_g f - f(0)g(0),$$

where the multiplication operator M_g is defined by $(M_g f)(z) = g(z)f(z)$.

In [24] Pommerenke introduced the operator J_g and showed that J_g is a bounded operator on the Hardy space H^2 if and only if $g \in BMOA$. The operators J_g and I_g , as well as their n -dimensional generalizations, acting on various spaces of analytic functions, have been recently studied, for example, in [1,4,5,12–18,20,21,26,29–33] (see also the related references therein). Some related operators can be also found in [6,19,27,28].

In this paper, we consider the products of composition operator and integral-type operators, which are defined by

$$C_\varphi J_g(f)(z) = \int_0^{\varphi(z)} f(\zeta) g'(\zeta) d\zeta, \quad C_\varphi I_g(f)(z) = \int_0^{\varphi(z)} f'(\zeta) g(\zeta) d\zeta \quad (1)$$

and

$$J_g C_\varphi(f)(z) = \int_0^z (f \circ \varphi)(\zeta) g'(\zeta) d\zeta, \quad I_g C_\varphi(f)(z) = \int_0^z (f \circ \varphi)'(\zeta) g(\zeta) d\zeta. \quad (2)$$

The boundedness and compactness of operators (1) and (2) between Bloch-type spaces and/or little Bloch-type spaces are studied. The study of these operators naturally comes from the isometry of some function spaces. Namely, it was shown in [11] that an operator T is a surjective isometry of the Dirichlet space

$$\mathcal{D}^p = \left\{ f \in H(\mathbb{D}) \mid \|f\|_{\mathcal{D}^p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p dm(z) < \infty \right\},$$

where $p \neq 2$, if and only if there is an automorphism ϕ of \mathbb{D} and unimodular constants λ_1 and λ_2 such that

$$(Tf)(z) = \lambda_1 f(0) + \lambda_2 \int_0^z (\phi'(\xi))^{2/p} f'(\phi(\xi)) d\xi \quad (3)$$

for every $f \in \mathcal{D}^p$. Let S^p be the space of all analytic functions f on \mathbb{D} such that $f' \in H^p$. An operator T is a surjective isometry of S^p with respect to the norm $\|f\|_{S^p}^p = |f(0)|^p + \|f'\|_{H^p}^p$ if and only if there is an automorphism ϕ of \mathbb{D} and unimodular constants λ_1 and λ_2 such that

$$(Tf)(z) = \lambda_1 f(0) + \lambda_2 \int_0^z (\phi'(\xi))^{1/p} f'(\phi(\xi)) d\xi \quad (4)$$

for every $f \in S^p$. Note that operators (3) and (4) are of type (2).

Throughout this paper, constants are denoted by C , they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $C^{-1}B \leq A \leq CB$.

2. Auxiliary results

In this section, we give some auxiliary results which are incorporated in the following lemmas. The following lemma is well known, see, for example, [23] or [29].

Lemma 1. Let $f \in \mathcal{B}^\alpha$. Then

$$|f(z)| \leq C \begin{cases} \|f\|_{\mathcal{B}^\alpha}, & \alpha \in (0, 1), \\ \|f\|_{\mathcal{B}^\alpha} \ln \frac{2}{1-|z|^2}, & \alpha = 1, \\ \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}}, & \alpha > 1, \end{cases}$$

for some $C > 0$ independent of f .

Let \mathcal{A}^1 denote the Bergman space, that is, the space of all $f \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f(z)|^2 dm(z) < \infty,$$

where $dm(z) = \frac{1}{\pi} r dr d\theta$ is the normalized area measure on \mathbb{D} .

The following result is well known [35].

Lemma 2. Suppose that $\alpha \in (0, \infty)$. Then, the following statements are true.

- (a) $(\mathcal{B}_0^\alpha)^* = \mathcal{A}^1$.
- (b) $(\mathcal{A}^1)^* = \mathcal{B}^\alpha$.
- (c) The second dual of \mathcal{B}_0^α is \mathcal{B}^α .

Lemma 3. (See [33].) Suppose that $\alpha \in (0, \infty) \setminus \{1\}$. Then there are two holomorphic maps $f_1, f_2 \in \mathcal{B}^\alpha$ with

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha (|f_1'(z)| + |f_2'(z)|) < \infty \quad (5)$$

and

$$\inf_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha-1} (|f_1(z)| + |f_2(z)|) > 0. \quad (6)$$

Based on a result from [25], in [10] the authors proved the following result.

Lemma 4. Suppose that $\alpha \in (0, \infty)$. Then, there exist two holomorphic maps $f_1, f_2 \in \mathcal{B}^\alpha$ such that

$$(1 - |z|^2)^\alpha (|f_1'(z)| + |f_2'(z)|) \asymp 1, \quad (7)$$

for all $z \in \mathbb{D}$.

The following lemma can be found in [23], when $\beta = 1$, see [22].

Lemma 5. A closed set K in \mathcal{B}_0^β is compact if and only if it is bounded and satisfies

$$\lim_{|z| \rightarrow 1} \sup_{f \in K} (1 - |z|^2)^\beta |f'(z)| = 0.$$

Remark 1. If in Lemma 5 we assume that K is not closed, then word *compact* can be replaced by *relatively compact*.

The next lemma characterizes the compactness of the operators in (1) and (2) in an usable way.

Lemma 6. The operator $C_\varphi J_g$ (respect. $C_\varphi I_g; I_g C_\varphi; J_g C_\varphi$): $\mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact if and only if $C_\varphi J_g$ (respect. $C_\varphi I_g; I_g C_\varphi; J_g C_\varphi$): $\mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in \mathcal{B}^α which converges to zero uniformly on compact subsets of \mathbb{D} , $C_\varphi J_g f_k \rightarrow 0$ (respect. $C_\varphi I_g f_k; I_g C_\varphi f_k; J_g C_\varphi f_k \rightarrow 0$) in \mathcal{B}^β as $k \rightarrow \infty$.

Proof. Assume the operator $C_\varphi J_g: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact and that $(f_k)_{k \in \mathbb{N}}$ is a sequence in \mathcal{B}^α such that $\sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{B}^\alpha} < \infty$ and $f_k \rightarrow 0$ uniformly on compact subsets of \mathbb{D} , as $k \rightarrow \infty$. By the compactness of $C_\varphi J_g: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ it follows that $C_\varphi J_g: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded and we have that the sequence $(C_\varphi J_g(f_k))_{k \in \mathbb{N}}$ has a subsequence $(C_\varphi J_g(f_{k_m}))_{m \in \mathbb{N}}$ which converges in \mathcal{B}^β , say, to f . In view of Lemma 1, it is clear that for any compact $K \subset \mathbb{D}$, there is a positive constant C_K such that

$$|C_\varphi J_g(f_{k_m})(z) - f(z)| \leq C_K \|C_\varphi J_g(f_{k_m}) - f\|_{\mathcal{B}^\beta}, \quad \text{for all } z \in K.$$

This implies that $C_\varphi J_g(f_{k_m})(z) - f(z) \rightarrow 0$ uniformly on compact subsets of \mathbb{D} , as $m \rightarrow \infty$. Since $f_{k_m} \rightarrow 0$ on compact subsets of \mathbb{D} , and by the following estimate

$$|C_\varphi J_g(f_{k_m})(z)| = \left| \int_0^{\varphi(z)} f_{k_m}(\zeta) g'(\zeta) d\zeta \right| \leq \max_{|\zeta| \leq |\varphi(z)|} |f_{k_m}(\zeta)| \max_{|\zeta| \leq |\varphi(z)|} |g'(\zeta)|$$

it is clear that for each $z \in \mathbb{D}$, $\lim_{m \rightarrow \infty} C_\varphi J_g(f_{k_m})(z) = 0$. Hence the limit function f is equal to 0. Since it holds for every subsequence of $(f_k)_{k \in \mathbb{N}}$ the implication follows.

Conversely, let $(h_k)_{k \in \mathbb{N}}$ be any sequence in the ball $\mathcal{K}_M = B_{\mathcal{B}^\alpha}(0, M)$ of \mathcal{B}^α . From the fact $\sup_{k \in \mathbb{N}} \|h_k\|_{\mathcal{B}^\alpha} \leq M < \infty$, we have that the sequence $(h_k)_{k \in \mathbb{N}}$ is uniformly bounded on compact subsets of \mathbb{D} and consequently normal by Montel's theorem. Hence we may extract a subsequence $(h_{k_j})_{j \in \mathbb{N}}$ which converges uniformly on compact subsets of \mathbb{D} to some $h \in H(\mathbb{D})$, moreover $h \in \mathcal{B}^\alpha$ and $\|h\|_{\mathcal{B}^\alpha} \leq M$. Thus, the sequence $(h_{k_j} - h)_{j \in \mathbb{N}}$ is such that $\|h_{k_j} - h\|_{\mathcal{B}^\alpha} \leq 2M < \infty$, and converges to 0 on compact subsets of \mathbb{D} as $j \rightarrow \infty$. By the hypothesis we have that $C_\varphi J_g(h_{k_j}) \rightarrow C_\varphi J_g(h)$ in \mathcal{B}^β . Thus the set $C_\varphi J_g(\mathcal{K}_M)$ is relatively compact, finishing the proof of the lemma for this case. The proofs in other cases are similar and are omitted. \square

For the case, $\alpha \in (0, 1)$ a slightly different result can be proved (see [23]).

Lemma 7. Assume that $\alpha \in (0, 1)$. Then the operator $C_\varphi J_g$ (respect. $C_\varphi I_g; I_g C_\varphi; J_g C_\varphi$) : $\mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact if and only if $C_\varphi J_g$ (respect. $C_\varphi I_g; I_g C_\varphi; J_g C_\varphi$) : $\mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in \mathcal{B}^α which converges to zero uniformly on \mathbb{D} , $C_\varphi J_g f_k \rightarrow 0$ (respect. $C_\varphi I_g f_k; I_g C_\varphi f_k; J_g C_\varphi f_k \rightarrow 0$) in \mathcal{B}^β as $k \rightarrow \infty$.

Lemma 8. Assume that $h \in H(\mathbb{D})$, $f \in \mathcal{B}^\alpha$, for some $\alpha > 0$, and that $z_0 \in \mathbb{D}$ is fixed. Then, the following statements are true.

(a) There is a positive constant C independent of f such that

$$\left| \int_0^{z_0} f(\zeta) h(\zeta) d\zeta \right| \leq C \|f\|_{\mathcal{B}^\alpha} \max_{|\zeta| \leq |z_0|} |h(\zeta)|.$$

(b) There is a positive constant C independent of f such that

$$\left| \int_0^{z_0} f'(\zeta) h(\zeta) d\zeta \right| \leq C \|f\|_{\mathcal{B}^\alpha} \max_{|\zeta| \leq |z_0|} |h(\zeta)|.$$

Proof. (a) We have

$$\begin{aligned} \left| \int_0^{z_0} f(\zeta) h(\zeta) d\zeta \right| &\leq \max_{|\zeta| \leq |z_0|} |f(\zeta)| \max_{|\zeta| \leq |z_0|} |h(\zeta)| = \max_{|\zeta| \leq |z_0|} \left| \int_0^\zeta f'(u) du + f(0) \right| \max_{|\zeta| \leq |z_0|} |h(\zeta)| \\ &\leq \left(|f(0)| + |z_0| \max_{|\zeta| \leq |z_0|} |f'(\zeta)| \right) \max_{|\zeta| \leq |z_0|} |h(\zeta)| \\ &= \left(|f(0)| + \frac{|z_0|}{(1 - |z_0|^2)^\alpha} \max_{|\zeta| = |z_0|} (1 - |z_0|^2)^\alpha |f'(\zeta)| \right) \max_{|\zeta| \leq |z_0|} |h(\zeta)| \\ &\leq \max \left\{ 1, \frac{|z_0|}{(1 - |z_0|^2)^\alpha} \right\} \|f\|_{\mathcal{B}^\alpha} \max_{|\zeta| \leq |z_0|} |h(\zeta)|, \end{aligned}$$

as desired.

(b) We have

$$\begin{aligned} \left| \int_0^{z_0} f'(\zeta) h(\zeta) d\zeta \right| &\leq |z_0| \max_{|\zeta| \leq |z_0|} |f'(\zeta)| \max_{|\zeta| \leq |z_0|} |h(\zeta)| \\ &= \frac{|z_0|}{(1 - |z_0|^2)^\alpha} \max_{|\zeta| = |z_0|} (1 - |z_0|^2)^\alpha |f'(\zeta)| \max_{|\zeta| \leq |z_0|} |h(\zeta)| \\ &\leq \frac{|z_0|}{(1 - |z_0|^2)^\alpha} \|f\|_{\mathcal{B}^\alpha} \max_{|\zeta| \leq |z_0|} |h(\zeta)|, \end{aligned}$$

finishing the proof of the lemma. \square

3. The boundedness and compactness of $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ (or \mathcal{B}_0^β)

In this section, we characterize the boundedness and compactness of the operator $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ (or \mathcal{B}_0^β).

Theorem 1. Let φ be an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. If $\alpha \in (0, 1)$, then the following statements hold.

(a) $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is bounded if and only if

$$M := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| < \infty. \quad (8)$$

(b) $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}_0^\beta$ is bounded if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| = 0. \quad (9)$$

Proof. (a) Assume that $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is bounded. From (1) we see that

$$(C_\varphi J_g f)'(z) = f(\varphi(z)) g'(\varphi(z)) \varphi'(z). \quad (10)$$

Choose $f_0(z) \equiv 1$. It is clear that $f_0 \in \mathcal{B}_0^\alpha$ and that $\|f_0\|_{\mathcal{B}^\alpha} = 1$. The boundedness of $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ implies that

$$\begin{aligned} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| &= (1 - |z|^2)^\beta |(C_\varphi J_g f_0)'(z)| \\ &\leq \|C_\varphi J_g\| \|f_0\|_{\mathcal{B}^\alpha} = \|C_\varphi J_g\| < \infty \end{aligned} \quad (11)$$

for any $z \in \mathbb{D}$. Therefore, we obtain (8), as desired.

Now assume that (8) holds. Then, by Lemma 1 and (10) we have

$$(1 - |z|^2)^\beta |(C_\varphi J_g f)'(z)| \leq C (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \|f\|_{\mathcal{B}^\alpha}. \quad (12)$$

From Lemma 8 (a) with $h = g'$ and $z_0 = \varphi(0)$, we have that

$$|(C_\varphi J_g f)(0)| = \left| \int_0^{\varphi(0)} f(\zeta) g'(\zeta) d\zeta \right| \leq C \|f\|_{\mathcal{B}^\alpha} \max_{|\zeta| \leq |\varphi(0)|} |g'(\zeta)|. \quad (13)$$

Since $|\varphi(0)| < 1$, it follows that $\max_{|\zeta| \leq |\varphi(0)|} |g'(\zeta)| < \infty$. From this and by taking the supremum in (12) over $z \in \mathbb{D}$, we obtain

$$\|C_\varphi J_g(f)\|_{\mathcal{B}^\beta} \leq C \left(\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| + \max_{|\zeta| \leq |\varphi(0)|} |g'(\zeta)| \right) \|f\|_{\mathcal{B}^\alpha},$$

which in view of (8) and (13) implies the boundedness of $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$.

(b) Assume that $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}_0^\beta$ is bounded. Let $f_0 \equiv 1$, then $C_\varphi J_g(f_0) \in \mathcal{B}_0^\beta$, that is, (9) holds, as desired.

Now, assume (9) holds. Let $f \in \mathcal{B}^\alpha$, then from (12) we see that (9) implies $C_\varphi J_g(f) \in \mathcal{B}_0^\beta$ for each $f \in \mathcal{B}^\alpha$. Moreover, (9) implies (8), so by (a) the operator $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded. Therefore, $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded too. \square

Theorem 2. Let φ be an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. If $\alpha \in (0, 1)$, then

(a) $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is compact if and only if (8) holds.

Also, the following statements are equivalent:

- (b) $C_\varphi J_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact;
- (c) $C_\varphi J_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is weakly compact;
- (d) Condition (9) holds;
- (e) $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact.

Proof. (a) Assume that $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is compact, then it is bounded and by Theorem 1 it follows that condition (8) holds.

Conversely, suppose that (8) holds. By Theorem 1, we know that $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is bounded. By Lemma 7, we should prove that $\|C_\varphi J_g f_k\|_{\mathcal{B}^\beta} \rightarrow 0$ as $k \rightarrow \infty$ for each sequence $(f_k)_{k \in \mathbb{N}} \subset \mathcal{B}^\alpha$ (or \mathcal{B}_0^α), such that $\sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{B}^\alpha} < \infty$ and

which converges to zero uniformly on $\overline{\mathbb{D}}$. We have

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup_{z \in \overline{\mathbb{D}}} (1 - |z|^2)^\beta |(C_\varphi J_g f_k)'(z)| &= \lim_{k \rightarrow \infty} \sup_{z \in \overline{\mathbb{D}}} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| |f_k(\varphi(z))| \\ &\leq \sup_{z \in \overline{\mathbb{D}}} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \lim_{k \rightarrow \infty} \|f_k\|_\infty = 0. \end{aligned}$$

On the other hand, we have

$$|(C_\varphi J_g f_k)(0)| = \left| \int_0^{\varphi(0)} f_k(\zeta) g'(\zeta) d\zeta \right| \leq C \|f_k\|_\infty \max_{|\zeta| \leq |\varphi(0)|} |g'(\zeta)| \rightarrow 0, \quad (14)$$

as $k \rightarrow \infty$. From last two estimates the compactness follows.

(b) \Rightarrow (c). By the definition every compact operator is weakly compact.

(c) \Rightarrow (d). It is obvious that $C_\varphi J_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded. Since $f_0(z) \equiv 1$ belongs to \mathcal{B}_0^α , we have that $C_\varphi J_g(1) \in \mathcal{B}_0^\beta$, that is, (9) holds.

(d) \Rightarrow (e). Condition (9) implies (8). Hence the set $C_\varphi J_g(\{f : \|f\|_{\mathcal{B}^\alpha} \leq 1\})$ is bounded in \mathcal{B}^β . Moreover, from (12) it follows that the set is bounded in \mathcal{B}_0^β . Taking the supremum in inequality (12) over the unit ball in \mathcal{B}^α , then letting $|z| \rightarrow 1$, applying (9) and Lemma 5, we obtain that the implication is true.

(e) \Rightarrow (b). This implication is obvious. \square

Next, we consider the case of $\alpha = 1$.

Theorem 3. Let φ be an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements hold.

(a) $C_\varphi J_g : \mathcal{B}$ (or \mathcal{B}_0) $\rightarrow \mathcal{B}^\beta$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} < \infty. \quad (15)$$

(b) $C_\varphi J_g : \mathcal{B}_0 \rightarrow \mathcal{B}_0^\beta$ is bounded if and only if conditions (9) and (15) hold.

Proof. (a) First, assume $C_\varphi J_g : \mathcal{B}$ (or \mathcal{B}_0) $\rightarrow \mathcal{B}^\beta$ is bounded. For $w \in \mathbb{D}$, set

$$f_w(z) = \ln \frac{2}{1 - \bar{w}z}.$$

It is easy to see that $f_w \in \mathcal{B}_0$ and $\sup_{w \in \mathbb{D}} \|f_w\|_{\mathcal{B}} \leq 2 + \ln 2$. Therefore

$$\begin{aligned} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} &= (1 - |z|^2)^\beta |(C_\varphi J_g f_{\varphi(z)})'(z)| \\ &\leq \|C_\varphi J_g f_{\varphi(z)}\|_{\mathcal{B}^\beta} \\ &\leq \|C_\varphi J_g\| \|f_{\varphi(z)}\|_{\mathcal{B}} < \infty. \end{aligned} \quad (16)$$

Taking the supremum in (16) over $z \in \mathbb{D}$, we obtain (15).

Conversely, assume that (15) holds. By Lemma 1 and (10) we have

$$(1 - |z|^2)^\beta |(C_\varphi J_g f)'(z)| \leq C \|f\|_{\mathcal{B}} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2}. \quad (17)$$

From (17) and (13) with $\alpha = 1$, the boundedness of $C_\varphi J_g : \mathcal{B}$ (or \mathcal{B}_0) $\rightarrow \mathcal{B}^\beta$ follows.

(b) If $C_\varphi J_g : \mathcal{B}_0 \rightarrow \mathcal{B}_0^\beta$ is bounded, then by (a) we see that (15) holds. By taking the function given by $f(z) \equiv 1$, we obtain (9).

Now, suppose that (9) and (15) hold. Then for each polynomial p the following inequality holds

$$\begin{aligned} (1 - |z|^2)^\beta |(C_\varphi J_g p)'(z)| &= (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| |p(\varphi(z))| \\ &\leq \|p\|_\infty (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)|. \end{aligned}$$

From this and (9), we obtain that for each polynomial p , $C_\varphi J_g(p) \in \mathcal{B}_0^\beta$. The set of all polynomials is dense in \mathcal{B}_0 , thus for every $f \in \mathcal{B}_0$ there is a sequence of polynomials $(p_k)_{k \in \mathbb{N}}$ such that $\|p_k - f\|_{\mathcal{B}} \rightarrow 0$ as $k \rightarrow \infty$. Hence

$$\|C_\varphi J_g p_k - C_\varphi J_g f\|_{\mathcal{B}^\beta} \leq \|C_\varphi J_g\| \|p_k - f\|_{\mathcal{B}} \rightarrow 0, \quad \text{as } k \rightarrow \infty,$$

since, as we have already proved, the operator $C_\varphi J_g : \mathcal{B}_0 \rightarrow \mathcal{B}^\beta$ is bounded. Hence $C_\varphi J_g(\mathcal{B}_0) \subset \mathcal{B}_0^\beta$. Since \mathcal{B}_0^β is closed subset of \mathcal{B}^β , $C_\varphi J_g : \mathcal{B}_0 \rightarrow \mathcal{B}_0^\beta$ is bounded. \square

Theorem 4. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements are equivalent:

- (a) $C_\varphi J_g : \mathcal{B} \rightarrow \mathcal{B}^\beta$ is compact and condition (9) holds;
- (b) $C_\varphi J_g : \mathcal{B}_0 \rightarrow \mathcal{B}_0^\beta$ is compact;
- (c) $C_\varphi J_g : \mathcal{B}_0 \rightarrow \mathcal{B}_0^\beta$ is weakly compact;
- (d) Condition (9) holds and

$$\lim_{|\varphi(z)| \rightarrow 1} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} = 0; \quad (18)$$

- (e) $C_\varphi J_g : \mathcal{B} \rightarrow \mathcal{B}_0^\beta$ is compact;
- (f) $C_\varphi J_g : \mathcal{B} \rightarrow \mathcal{B}_0^\beta$ is bounded.

Proof. (d) \Rightarrow (a). Clearly (9) implies (8). From (18) we see that there is an $r_0 \in (0, 1)$ such that

$$(1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} < \varepsilon$$

for every $|\varphi(z)| > r_0$. Let $(f_k)_{k \in \mathbb{N}}$ be a norm bounded sequence in \mathcal{B} such that $f_k \rightarrow 0$ on compact subsets of \mathbb{D} as $k \rightarrow \infty$. By Lemma 1, we obtain

$$\begin{aligned} (1 - |z|^2)^\beta |(C_\varphi J_g f_k)'(z)| &= |f_k(\varphi(z))| (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \\ &\leq \sup_{|\varphi(z)| \leq r_0} |f_k(\varphi(z))| \sup_{|\varphi(z)| \leq r_0} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \\ &\quad + C \|f_k\|_{\mathcal{B}} \sup_{|\varphi(z)| > r_0} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} \\ &\leq M \sup_{|\zeta| \leq r_0} |f_k(\zeta)| + \varepsilon C \|f_k\|_{\mathcal{B}}. \end{aligned} \quad (19)$$

We also have that

$$|(C_\varphi J_g f_k)(0)| = \left| \int_0^{\varphi(0)} f_k(\zeta) g'(\zeta) d\zeta \right| \leq \max_{|\zeta| \leq |\varphi(0)|} |f_k(\zeta)| \max_{|\zeta| \leq |\varphi(0)|} |g'(\zeta)| \rightarrow 0, \quad (20)$$

as $k \rightarrow \infty$.

Taking the supremum over $z \in \mathbb{D}$ and letting $k \rightarrow \infty$ in (19) and (20), we obtain that $\|C_\varphi J_g f_k\|_{\mathcal{B}^\beta} \rightarrow 0$ as $k \rightarrow \infty$. Hence, the operator $C_\varphi J_g : \mathcal{B} \rightarrow \mathcal{B}^\beta$ is compact.

(a) \Rightarrow (b). Assume that $C_\varphi J_g : \mathcal{B} \rightarrow \mathcal{B}^\beta$ is compact and (9) holds. As in Theorem 3, for each polynomial p we have that $C_\varphi J_g(p) \in \mathcal{B}_0^\beta$. Because the polynomials are dense in \mathcal{B}_0 and $\mathcal{B}_0^{**} = \mathcal{B}$, it follows that the polynomials are w^* -dense in \mathcal{B} . Thus, for each $f \in \mathcal{B}$ there is a sequence of polynomials $(p_m)_{m \in \mathbb{N}}$, such that $\sup_{m \in \mathbb{N}} \|p_m\|_{\mathcal{B}} < \infty$ and $p_m \rightarrow f$ uniformly on compact subsets of \mathbb{D} as $m \rightarrow \infty$. By the compactness, we have that there is a subsequence $(p_{m_k})_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} \|C_\varphi J_g(p_{m_k}) - C_\varphi J_g(f)\|_{\mathcal{B}^\beta} = 0,$$

which implies that $C_\varphi J_g(\mathcal{B}) \subset \mathcal{B}_0^\beta$. Hence, the image of the unit ball of \mathcal{B} under the operator $C_\varphi J_g$ is relatively compact in \mathcal{B}_0^β , which implies that $C_\varphi J_g : \mathcal{B}_0 \rightarrow \mathcal{B}_0^\beta$ is compact.

(b) \Rightarrow (c). This implication is clear.

(c) \Rightarrow (d). By putting $f(z) \equiv 1$, (9) follows. By Lemma 2 we know that $(\mathcal{B}_0^\alpha)^{**} = \mathcal{B}^\alpha$. Since $C_\varphi J_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ and $(\mathcal{B}_0^\beta)^* = (\mathcal{B}_0^\alpha)^* = \mathcal{A}^1$, we have that $(C_\varphi J_g)^* : \mathcal{A}^1 \rightarrow \mathcal{A}^1$. Hence every bounded linear functional \mathcal{L} on \mathcal{B}_0^β can be identified by a function $h \in \mathcal{A}^1$, so that for every $f \in \mathcal{B}_0^\alpha$ and $h \in \mathcal{A}^1$, we have

$$\langle C_\varphi J_g(f), h \rangle = \langle f, (C_\varphi J_g)^*(h) \rangle.$$

On the other hand, by Lemma 2 we have $(\mathcal{A}^1)^* = \mathcal{B}^\alpha$, which implies that $(C_\varphi J_g)^{**} : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$. Hence every $f \in \mathcal{B}_0^\alpha$ can be viewed as an element of the space $(\mathcal{A}^1)^*$ and

$$\langle f, (C_\varphi J_g)^*(h) \rangle = \langle (C_\varphi J_g)^{**}(f), h \rangle.$$

From these two equalities we have that

$$\langle C_\varphi J_g(f), h \rangle = \langle (C_\varphi J_g)^{**}(f), h \rangle,$$

for every $h \in \mathcal{A}^1$. By a known consequence of Hahn–Banach theorem we obtain $(C_\varphi J_g)^{**}(f) = C_\varphi J_g(f)$ for every $f \in \mathcal{B}_0^\alpha$. Since \mathcal{B}_0^α is w^* dense in \mathcal{B}^α , it follows that $(C_\varphi J_g)^{**}(f) = C_\varphi J_g(f)$ for every $f \in \mathcal{B}^\alpha$. From this and by Gantmacher's theorem we have that $C_\varphi J_g(\mathcal{B}^\alpha) \subset \mathcal{B}_0^\beta$. Note that α is an arbitrary positive number here.

Now assume that the condition (18) does not hold. If it were, then it would exist an $\varepsilon_0 > 0$ and a sequence $(\varphi(z_k))_{k \in \mathbb{N}} \subset \mathbb{D}$, such that $\lim_{k \rightarrow \infty} |\varphi(z_k)| = 1$, and

$$(1 - |z_k|^2)^\beta |g'(\varphi(z_k))| |\varphi'(z_k)| \ln \frac{2}{1 - |\varphi(z_k)|^2} \geq \varepsilon_0 > 0$$

for sufficiently large k . We may also assume that

$$\frac{1 - |\varphi(z_{k-1})|}{2} > 1 - |\varphi(z_k)|, \quad k \in \mathbb{N}.$$

Then, for every non-negative integer s there is at most one $\varphi(z_k)$ such that $1 - \frac{1}{2^s} \leq |\varphi(z_k)| < 1 - \frac{1}{2^{s+1}}$. Hence, there is $M \in \mathbb{N}$ such that for any Carleson window

$$Q = \{re^{i\theta} \mid 0 < 1 - r < l(Q), \quad |\theta - \theta_0| < l(Q)\}$$

and $s \in \mathbb{N}$, there is at most M elements in the following set

$$\{\varphi(z_k) \in Q \mid 2^{-(s+1)}l(Q) < 1 - |\varphi(z_k)| < 2^{-s}l(Q)\}.$$

Therefore, $(\varphi(z_k))_{k \in \mathbb{N}}$ is an interpolating sequence for \mathcal{B} , in sense of [3].

By [3] we have that there is a function $h \in \mathcal{B}$ such that

$$h(\varphi(z_k)) = \ln \frac{2}{1 - |\varphi(z_k)|^2}, \quad k \in \mathbb{N}.$$

We have

$$\begin{aligned} (1 - |z_k|^2)^\beta |(C_\varphi J_g h)'(z_k)| &= (1 - |z_k|^2)^\beta |g'(\varphi(z_k))| |\varphi'(z_k)| |h(\varphi(z_k))| \\ &= (1 - |z_k|^2)^\beta |g'(\varphi(z_k))| |\varphi'(z_k)| \ln \frac{2}{1 - |\varphi(z_k)|^2} \geq \varepsilon_0. \end{aligned}$$

Thus, $C_\varphi J_g(h) \notin \mathcal{B}_0^\beta$, which is a contradiction.

(e) \Rightarrow (b). This implication is obvious.

(d) \Rightarrow (e). Suppose that (9) and (18) hold. By (18), we have that for every $\varepsilon > 0$, there exists an $r \in (0, 1)$, such that

$$(1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} < \varepsilon$$

when $r < |\varphi(z)| < 1$. By (9), there exists a $\sigma \in (0, 1)$ such that

$$|g'(\varphi(z))| |\varphi'(z)| (1 - |z|^2)^\beta < \varepsilon / \ln \frac{2}{1 - r^2}$$

when $\sigma < |z| < 1$.

Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have

$$(1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} < \varepsilon. \quad (21)$$

On the other hand, if $|\varphi(z)| \leq r$ and $\sigma < |z| < 1$, we have

$$(1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} < (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - r^2} < \varepsilon. \quad (22)$$

Combining (21) with (22), we obtain

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} = 0. \quad (23)$$

By Lemma 1, we have

$$(1 - |z|^2)^\beta |(C_\varphi J_g f)'(z)| \leq C \|f\|_{\mathcal{B}} (1 - |z|^2)^\beta |g'(\varphi(z))| |\varphi'(z)| \ln \frac{2}{1 - |\varphi(z)|^2}. \quad (24)$$

From (23), condition (15) follows. Hence the set $C_\varphi J_g(\{f: \|f\|_{\mathcal{B}} \leq 1\})$ is bounded in \mathcal{B}^β . Moreover, from (23) it follows that the set is bounded in \mathcal{B}_0^β . Taking the supremum in (24) over all $f \in \mathcal{B}$ such that $\|f\|_{\mathcal{B}} \leq 1$, then letting $|z| \rightarrow 1$, employing (23) and Lemma 5, we obtain the desired result.

Finally note that the implication (e) \Rightarrow (f) is obvious, and that (f) \Rightarrow (d) follows from the proof of (c) \Rightarrow (d). \square

Theorem 5. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the operator $C_\varphi J_g : \mathcal{B} \rightarrow \mathcal{B}^\beta$ is compact if and only if it is bounded and condition (18) holds.

Proof. Sufficiency. Since $C_\varphi J_g : \mathcal{B} \rightarrow \mathcal{B}^\beta$ is bounded, by taking $f_0(z) = 1$, we see that (8) holds. The rest of the proof is the same as the proof of Theorem 4 (d) \Rightarrow (a) and is omitted.

Necessity. Assume that $(z_k)_{k \in \mathbb{N}}$ is a sequence in \mathbb{D} such that $\lim_{k \rightarrow \infty} |\varphi(z_k)| = 1$ (if such a sequence does not exist then (18) is vacuously satisfied). Let

$$f_k(z) = \left(\ln \frac{2}{1 - |\varphi(z_k)|^2} \right)^{-1} \left(\ln \frac{2}{1 - \overline{\varphi(z_k)}z} \right)^2, \quad k \in \mathbb{N}.$$

By some simple calculation, we find that $\sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{B}} \leq C$. Moreover $f_k \rightarrow 0$ uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$. Since $C_\varphi J_g : \mathcal{B} \rightarrow \mathcal{B}^\beta$ is compact, by Lemma 6, we have $\lim_{k \rightarrow \infty} \|C_\varphi J_g f_k\|_{\mathcal{B}^\beta} = 0$. From this and since

$$\|C_\varphi J_g f_k\|_{\mathcal{B}^\beta} \geq \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |(C_\varphi J_g f_k)'(z)| \geq (1 - |z_k|^2)^\beta |\varphi'(z_k)| |g'(\varphi(z_k))| \ln \frac{2}{1 - |\varphi(z_k)|^2},$$

we have that

$$\lim_{k \rightarrow \infty} (1 - |z_k|^2)^\beta |\varphi'(z_k)| |g'(\varphi(z_k))| \ln \frac{2}{1 - |\varphi(z_k)|^2} = 0,$$

which implies that (18) holds. \square

Theorem 6. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$ and $\alpha > 1$. Then

(a) $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} |g'(\varphi(z))| < \infty. \quad (25)$$

(b) $C_\varphi J_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded if and only if conditions (9) and (25) hold.

Proof. (a) Assume $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is bounded. For $w \in \mathbb{D}$, let

$$f_w(z) = \frac{1}{(1 - \bar{w}z)^{\alpha-1}}.$$

It is easy to check that $f_w \in \mathcal{B}_0^\alpha$ and $\sup_{w \in \mathbb{D}} \|f_w\|_{\mathcal{B}^\alpha} \leq (\alpha - 1)2^\alpha + 1$. The boundedness of $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ implies

$$\begin{aligned} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} |g'(\varphi(z))| &= (1 - |z|^2)^\beta |(C_\varphi J_g f_{\varphi(z)})'(z)| \\ &\leq \|C_\varphi J_g\| \|f_{\varphi(z)}\|_{\mathcal{B}^\alpha} \\ &\leq ((\alpha - 1)2^\alpha + 1) \|C_\varphi J_g\| < \infty. \end{aligned} \quad (26)$$

By taking the supremum over $z \in \mathbb{D}$ we obtain (25), as desired.

Now, assume that (25) holds. Then, by Lemma 1 we have

$$(1 - |z|^2)^\beta |(C_\varphi J_g f)'(z)| \leq C \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} |g'(\varphi(z))| \|f\|_{\mathcal{B}^\alpha}. \quad (27)$$

From (13) and by taking the supremum in (27) over $z \in \mathbb{D}$, we obtain that $C_\varphi J_g : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is bounded.

(b) We omit the proof since it is similar to the proof of Theorem 3(b). \square

Theorem 7. Let φ be an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. If $\alpha > 1$, then the following statements are equivalent.

- (a) $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact and condition (9) holds;
- (b) $C_\varphi J_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact;
- (c) $C_\varphi J_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is weakly compact;

(d) Condition (9) holds and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} |g'(\varphi(z))| = 0; \quad (28)$$

(e) $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact;

(f) $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded.

Proof. (d) \Rightarrow (a). Assume that condition (d) holds. Then, condition (8) holds and there is an $r_0 \in (0, 1)$ such that

$$\frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} |g'(\varphi(z))| < \varepsilon$$

whenever $|\varphi(z)| > r_0$. Let $(f_k)_{k \in \mathbb{N}}$ be a bounded sequence in \mathcal{B}^α such that $f_k \rightarrow 0$ on compacts of \mathbb{D} as $k \rightarrow \infty$. By Lemma 1, it follows that

$$\begin{aligned} (1 - |z|^2)^\beta |(C_\varphi J_g f_k)'(z)| &= (1 - |z|^2)^\beta |\varphi'(z)| |g'(\varphi(z))| |f_k(\varphi(z))| \\ &\leq \sup_{|\varphi(z)| \leq r_0} |f_k(\varphi(z))| \sup_{|\varphi(z)| \leq r_0} (1 - |z|^2)^\beta |\varphi'(z)| |g'(\varphi(z))| \\ &\quad + C \|f_k\|_{\mathcal{B}^\alpha} \sup_{|\varphi(z)| > r_0} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} |g'(\varphi(z))| \\ &\leq M \sup_{|\varphi(z)| \leq r_0} |f_k(\varphi(z))| + \varepsilon C \|f_k\|_{\mathcal{B}^\alpha}. \end{aligned} \quad (29)$$

We also have that (20) holds. Taking the supremum over $z \in \mathbb{D}$ in (29), then letting $k \rightarrow \infty$, we obtain that $\lim_{k \rightarrow \infty} \|C_\varphi J_g f_k\|_{\mathcal{B}^\beta} = 0$. Hence, the operator $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact.

(a) \Rightarrow (b). The proof is similar to the corresponding proof of Theorem 4 and will be omitted.

(b) \Rightarrow (c). This statement is clear.

(c) \Rightarrow (d). By putting $f(z) \equiv 1$, (9) follows. Since $C_\varphi J_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is weakly compact, then similarly to the proof of Theorem 4, we see that $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded. Assume that condition (28) does not hold. If it were, then it would exist $\varepsilon_0 > 0$ and a sequence $(\varphi(z_k))_{k \in \mathbb{N}} \subset \mathbb{D}$ satisfying the condition $\lim_{k \rightarrow \infty} |\varphi(z_k)| = 1$, such that

$$\frac{(1 - |z_k|^2)^\beta |\varphi'(z_k)|}{(1 - |\varphi(z_k)|^2)^{\alpha-1}} |g'(\varphi(z_k))| \geq \varepsilon_0 > 0$$

for sufficiently large $k \in \mathbb{N}$. According to Lemma 3 there are functions $f_1, f_2 : \mathbb{D} \rightarrow \mathbb{C}$ with (5) and (6). We have $C_\varphi J_g f_1, C_\varphi J_g f_2 \in \mathcal{B}_0^\beta$. On the other hand, we have

$$\begin{aligned} (1 - |z_k|^2)^\beta &(|(C_\varphi J_g f_1)'(z_k)| + |(C_\varphi J_g f_2)'(z_k)|) \\ &= (1 - |z_k|^2)^\beta (|f_1(\varphi(z_k))| + |f_2(\varphi(z_k))|) |g'(\varphi(z_k))| |\varphi'(z_k)| \\ &\geq C \frac{(1 - |z_k|^2)^\beta |\varphi'(z_k)|}{(1 - |\varphi(z_k)|^2)^{\alpha-1}} |g'(\varphi(z_k))| \geq C \varepsilon_0, \end{aligned}$$

arriving at a contradiction.

(e) \Rightarrow (b). This implication is obvious.

(d) \Rightarrow (e). Similar to the proof of Theorem 4, from (9) and (28) we obtain that

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} |g'(\varphi(z))| = 0. \quad (30)$$

By Lemma 1 we have

$$(1 - |z|^2)^\beta |(C_\varphi J_g f)'(z)| \leq C \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} |g'(\varphi(z))| \|f\|_{\mathcal{B}^\alpha}. \quad (31)$$

From (30) we have that (25) holds. Hence the set $C_\varphi J_g(\{f : \|f\|_{\mathcal{B}^\alpha} \leq 1\})$ is bounded in \mathcal{B}^β . Moreover, from (31) it follows that the set is bounded in \mathcal{B}_0^β . Taking the supremum in (31) over the unit ball in \mathcal{B}^α , then letting $|z| \rightarrow 1$, applying (30) and Lemma 5, the implication follows.

(e) \Rightarrow (f). This implication is obvious.

(f) \Rightarrow (d). The implication follows from (c) \Rightarrow (d). \square

Theorem 8. Assume that $\alpha > 1$ and φ is an analytic self-map of \mathbb{D} and $g \in H(\mathbb{D})$. Then $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact if and only if it is bounded and condition (28) holds.

Proof. First, assume that $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded and (28) holds, then the proof is similar to (d) \Rightarrow (a) of Theorem 7 and is omitted.

Now assume $C_\varphi J_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact. Let $(z_k)_{k \in \mathbb{N}}$ be a sequence of \mathbb{D} such that $\lim_{k \rightarrow \infty} |\varphi(z_k)| = 1$. Set

$$f_k(z) = \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)z})^\alpha}, \quad k \in \mathbb{N}. \quad (32)$$

It is easy to see that $\sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{B}^\alpha} < \infty$, $f_k \in \mathcal{B}_0^\alpha$, $k \in \mathbb{N}$ and $f_k \rightarrow 0$ uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$. The rest of the proof is similar to the proof of Theorem 5, hence is omitted. \square

By using the same methods as in the proofs of Theorems 1–8 we can prove the following results. Their proofs will be omitted.

Theorem 9. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. If $\alpha \in (0, 1)$, then the following statements hold.

- (a) $J_g C_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is bounded if and only if $J_g C_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is compact if and only if $g \in \mathcal{B}^\beta$.
- (b) $J_g C_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}_0^\beta$ is bounded if and only if $J_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact if and only if $J_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is weakly compact if and only if $J_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact if and only if $g \in \mathcal{B}_0^\beta$.

Theorem 10. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements hold.

- (a) $J_g C_\varphi : \mathcal{B}$ (or \mathcal{B}_0) $\rightarrow \mathcal{B}^\beta$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |g'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} < \infty.$$

- (b) $J_g C_\varphi : \mathcal{B}_0 \rightarrow \mathcal{B}_0^\beta$ is bounded if and only if $J_g C_\varphi : \mathcal{B}_0 \rightarrow \mathcal{B}^\beta$ is bounded and $g \in \mathcal{B}_0^\beta$.

Theorem 11. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then

- (i) $J_g C_\varphi : \mathcal{B} \rightarrow \mathcal{B}^\beta$ is compact if and only if $J_g C_\varphi : \mathcal{B} \rightarrow \mathcal{B}^\beta$ is bounded and

$$\lim_{|\varphi(z)| \rightarrow 1} (1 - |z|^2)^\beta |g'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} = 0;$$

- (ii) The following statements are equivalent:

- (a) $J_g C_\varphi : \mathcal{B} \rightarrow \mathcal{B}^\beta$ is compact and $g \in \mathcal{B}_0^\beta$;
- (b) $J_g C_\varphi : \mathcal{B}_0 \rightarrow \mathcal{B}_0^\beta$ is compact;
- (c) $J_g C_\varphi : \mathcal{B}_0 \rightarrow \mathcal{B}_0^\beta$ is weakly compact;
- (d) $g \in \mathcal{B}_0^\beta$ and

$$\lim_{|\varphi(z)| \rightarrow 1} (1 - |z|^2)^\beta |g'(z)| \ln \frac{2}{1 - |\varphi(z)|^2} = 0;$$

- (e) $J_g C_\varphi : \mathcal{B} \rightarrow \mathcal{B}_0^\beta$ is compact;
- (f) $J_g C_\varphi : \mathcal{B} \rightarrow \mathcal{B}_0^\beta$ is bounded.

Theorem 12. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. If $\alpha > 1$, then the following statements hold.

- (a) $J_g C_\varphi : \mathcal{B}^\alpha$ (or \mathcal{B}_0^α) $\rightarrow \mathcal{B}^\beta$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |g'(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} < \infty.$$

- (b) $J_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded if and only if $J_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}^\beta$ is bounded and $g \in \mathcal{B}_0^\beta$.

Theorem 13. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. If $\alpha > 1$, then

- (i) $J_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact if and only if $J_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |g'(z)|}{(1 - |\varphi(z)|^2)^{\alpha-1}} = 0. \quad (33)$$

(ii) The following statements are equivalent:

- (a) $J_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact and $g \in \mathcal{B}_0^\beta$;
- (b) $J_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact;
- (c) $J_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is weakly compact;
- (d) $g \in \mathcal{B}_0^\beta$ and (33) holds;
- (e) $J_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact;
- (f) $J_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded.

4. The boundedness and compactness of $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$

In this section, we consider the boundedness and compactness of $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$.

Theorem 14. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. If $0 < \alpha, \beta < \infty$, then the following statements are equivalent.

- (a) $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded;
- (b) $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}^\beta$ is bounded;

$$(c) \quad \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} |g(\varphi(z))| < \infty. \quad (34)$$

Proof. (c) \Rightarrow (a). From (1) we have that $(C_\varphi I_g f)'(z) = \varphi'(z)g(\varphi(z))f'(\varphi(z))$. Hence, for $f \in \mathcal{B}^\alpha$ it follows that

$$(1 - |z|^2)^\beta |(C_\varphi I_g f)'(z)| = \frac{(1 - |z|^2)^\beta |\varphi'(z)| |g(\varphi(z))|}{(1 - |\varphi(z)|^2)^\alpha} |f'(\varphi(z))| (1 - |\varphi(z)|^2)^\alpha \quad (35)$$

$$\leq \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} |g(\varphi(z))| \|f\|_{\mathcal{B}^\alpha}. \quad (36)$$

From this and the inequality

$$|(C_\varphi I_g f)(0)| = \left| \int_0^{\varphi(0)} f'(\zeta) g(\zeta) d\zeta \right| \leq C \|f\|_{\mathcal{B}^\alpha} \max_{|\zeta| \leq |\varphi(0)|} |g(\zeta)|,$$

we see that the operator $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded.

(a) \Rightarrow (b). This implication is obvious.

(b) \Rightarrow (c). Since $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}^\beta$ is bounded, then there exists a constant C such that $\|C_\varphi I_g f\|_{\mathcal{B}^\beta} \leq C \|f\|_{\mathcal{B}^\alpha}$ for all $f \in \mathcal{B}_0^\alpha$. Taking $f(z) = z$, we obtain that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |\varphi'(z)| |g(\varphi(z))| < \infty. \quad (37)$$

For $w \in \mathbb{D}$, when $\alpha = 1$, set $f_w(z) = \ln \frac{2}{1-wz}$, and when $\alpha \neq 1$, set $f_w(z) = \frac{1}{(1-wz)^{\alpha-1}}$. From Section 3, we know that $f_w \in \mathcal{B}_0^\alpha$ and $\sup_{w \in \mathbb{D}} \|f_w\|_{\mathcal{B}^\alpha} < \infty$. Hence, we have

$$\frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| |g(\varphi(z))| \leq C \|C_\varphi I_g f_{\varphi(z)}\|_{\mathcal{B}^\beta} \leq C \|C_\varphi I_g\| \|f_{\varphi(z)}\|_{\mathcal{B}^\alpha} < \infty. \quad (38)$$

In addition, we have

$$\sup_{|\varphi(z)| > 1/2} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| |g(\varphi(z))| \leq 2 \sup_{|\varphi(z)| > 1/2} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| |g(\varphi(z))|$$

and

$$\sup_{|\varphi(z)| \leq 1/2} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| |g(\varphi(z))| \leq \frac{4^\alpha}{3^\alpha} \sup_{|\varphi(z)| \leq 1/2} (1 - |z|^2)^\beta |\varphi'(z)| |g(\varphi(z))|.$$

Thus, from the last two inequalities, (37) and (38), we see that (34) holds. \square

Theorem 15. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. If $0 < \alpha, \beta < \infty$, then $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded if and only if

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} |g(\varphi(z))| = 0. \quad (39)$$

Proof. Suppose that (39) holds. From Theorem 14 we see that $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded. Therefore we only need to prove that $C_\varphi I_g(\mathcal{B}^\alpha) \subseteq \mathcal{B}_0^\beta$. Letting $|z| \rightarrow 1$ in (36), we have for every $f \in \mathcal{B}^\alpha$, $C_\varphi I_g f \in \mathcal{B}_0^\beta$, which is what we wanted to prove.

Conversely, we assume that $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded. For $0 < \alpha < \infty$, by Lemma 4, there exist functions $f, h \in \mathcal{B}^\alpha$ such that

$$|f'(z)| + |h'(z)| \geq \frac{C}{(1 - |z|)^\alpha}.$$

Hence

$$C \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|)^\alpha} |g(\varphi(z))| \leq (1 - |z|^2)^\beta |(C_\varphi I_g f)'(z)| + (1 - |z|^2)^\beta |(C_\varphi I_g h)'(z)|.$$

Since $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded, then $C_\varphi I_g f, C_\varphi I_g h \in \mathcal{B}_0^\beta$, and thus the right-hand side of the above inequality tends to zero as $|z| \rightarrow 1$. Hence (39) is satisfied. \square

Theorem 16. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$ and $\alpha, \beta \in (0, \infty)$. Then $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded if and only if $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded and

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |\varphi'(z)| |g(\varphi(z))| = 0. \quad (40)$$

Proof. Suppose $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded, then $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded. Set $f(z) = z \in \mathcal{B}_0^\alpha$, we see that the boundedness of $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ implies (40).

Conversely, assume that $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded and (40) holds. Then, similar to the proof of Theorem 3 we obtain the desired result. \square

Theorem 17. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$ and $\alpha, \beta \in (0, \infty)$. Then the following statements are equivalent.

- (a) $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact;
- (b) $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}^\beta$ is compact;
- (c) Condition (34) holds and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} |g(\varphi(z))| = 0. \quad (41)$$

Proof. (c) \Rightarrow (a). Let $(f_k)_{k \in \mathbb{N}}$ be a sequence in \mathcal{B}^α such that $\sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{B}^\alpha} < \infty$ and f_k converges to 0 uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$. By the assumption, for any $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} |g(\varphi(z))| < \varepsilon.$$

Let $K = \{z \in \mathbb{D} : |\varphi(z)| \leq \delta\}$. Then, we have that

$$\begin{aligned} \|C_\varphi I_g f_k\|_{\mathcal{B}^\beta} &= \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |\varphi'(z)| |f'_k(\varphi(z))| |g(\varphi(z))| + \left| \int_0^{\varphi(0)} f'_k(\zeta) g(\zeta) d\zeta \right| \\ &\leq \sup_{z \in K} (1 - |z|^2)^\beta |\varphi'(z)| |f'_k(\varphi(z))| |g(\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{D} \setminus K} (1 - |z|^2)^\beta |\varphi'(z)| |f'_k(\varphi(z))| |g(\varphi(z))| + \max_{|\zeta| \leq |\varphi(0)|} |f'_k(\zeta)| \max_{|\zeta| \leq |\varphi(0)|} |g(\zeta)| \\ &\leq G \sup_{|\zeta| \leq \delta} |f'_k(\zeta)| + \sup_{z \in \mathbb{D} \setminus K} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} |g(\varphi(z))| \|f_k\|_{\mathcal{B}^\alpha} \\ &\quad + \max_{|\zeta| \leq |\varphi(0)|} |f'_k(\zeta)| \max_{|\zeta| \leq |\varphi(0)|} |g(\zeta)|, \end{aligned} \quad (42)$$

where

$$G := \sup_{z \in K} (1 - |z|^2)^\beta |\varphi'(z)| |g(\varphi(z))| < \infty.$$

Since f_k converges to 0 uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$, Cauchy's estimate implies that $f'_k \rightarrow 0$ as $k \rightarrow \infty$ on compact subsets of \mathbb{D} , in particular, on $|\zeta| \leq \delta$ and $|\zeta| \leq |\varphi(0)|$. Hence, letting $k \rightarrow \infty$ in (42) we obtain $\lim_{k \rightarrow \infty} \|C_\varphi I_g f_k\|_{\mathcal{B}^\beta} = 0$. From this and applying Lemma 6 the result follows.

(a) \Rightarrow (b). This implication is clear.

(b) \Rightarrow (c). From Theorem 14, it is clear that (34) holds. Now we prove that (41) holds. Assume that $(z_k)_{k \in \mathbb{N}}$ is a sequence in \mathbb{D} such that $\lim_{k \rightarrow \infty} |\varphi(z_k)| = 1$ and let $(f_k)_{k \in \mathbb{N}}$ be defined as in (32). Since $\sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{B}^\alpha} < \infty$, $f_k \in \mathcal{B}_0^\alpha$, $k \in \mathbb{N}$ and $f_k \rightarrow 0$ uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$, in view of the compactness of $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}^\beta$, by Lemma 6, we have $\lim_{k \rightarrow \infty} \|C_\varphi I_g f_k\|_{\mathcal{B}^\beta} = 0$. From this and since

$$\|C_\varphi I_g f_k\|_{\mathcal{B}^\beta} \geq \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |(C_\varphi I_g f_k)'(z)| \geq (1 - |z_k|^2)^\beta |\varphi'(z_k)| |g(\varphi(z_k))| \frac{\alpha |\varphi(z_k)|}{(1 - |\varphi(z_k)|^2)^\alpha},$$

we have that

$$\lim_{k \rightarrow \infty} (1 - |z_k|^2)^\beta |\varphi'(z_k)| |g(\varphi(z_k))| \frac{|\varphi(z_k)|}{(1 - |\varphi(z_k)|^2)^\alpha} = 0,$$

which implies (41), completing the proof of the theorem. \square

Theorem 18. Assume that φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. If $0 < \alpha, \beta < \infty$, then the following statements are equivalent.

- (a) $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact;
- (b) $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact;
- (c) Condition (39) holds;
- (d) $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is weakly compact.

Proof. (b) \Rightarrow (a). This implication is obvious.

(c) \Rightarrow (b). Condition (39) implies that (34) holds, so by Theorem 14, the set $C_\varphi I_g(\{f : \|f\|_{\mathcal{B}^\alpha} \leq 1\})$ is bounded in \mathcal{B}^β . Moreover, (36) implies that the set is bounded in \mathcal{B}_0^β . Taking the supremum in (36) over all $f \in \mathcal{B}^\alpha$ such that $\|f\|_{\mathcal{B}^\alpha} \leq 1$, then letting $|z| \rightarrow 1$, we obtain

$$\lim_{|z| \rightarrow 1} \sup_{\|f\|_{\mathcal{B}^\alpha} \leq 1} (1 - |z|^2)^\beta |(C_\varphi I_g f)'(z)| = 0,$$

from which by Lemma 5 we obtain that $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact.

(a) \Rightarrow (c). Since $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact, then $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded and consequently by taking $f(z) = z$, we obtain (40). Also, $C_\varphi I_g : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}^\beta$ is compact, hence from Theorem 17 we obtain (41). From (40) and (41), similar to the proof of Theorem 4 (d) \Rightarrow (e), we obtain (39), as desired.

(a) \Rightarrow (d). This implication is clear.

(d) \Rightarrow (c). Similar to the proof of Theorem 4 (c) \Rightarrow (d), we see that $C_\varphi I_g : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded. From Theorem 15, (39) follows. \square

By using the same methods as in the proofs of Theorems 14–18 we can prove the following results.

Theorem 19. Assume that $\alpha, \beta \in (0, \infty)$, φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements are equivalent.

- (a) $I_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded;
- (b) $I_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}^\beta$ is bounded;
- (c)
$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |\varphi'(z)| |g(z)|}{(1 - |\varphi(z)|^2)^\alpha} < \infty.$$

Theorem 20. Assume that $\alpha, \beta \in (0, \infty)$, φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then the following statements are equivalent.

- (a) $I_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact;

- (b) $I_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}^\beta$ is compact;
 (c) $I_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\beta |\varphi'(z)| |g(z)|}{(1 - |\varphi(z)|^2)^\alpha} = 0.$$

Theorem 21. Assume that $\alpha, \beta \in (0, \infty)$, φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then $I_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded if and only if $I_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}^\beta$ is bounded and

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |\varphi'(z)| |g(z)| = 0.$$

Theorem 22. Assume that $\alpha, \beta \in (0, \infty)$, φ is an analytic self-map of the unit disk and $g \in H(\mathbb{D})$. Then $I_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is bounded if and only if $I_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact if and only if $I_g C_\varphi : \mathcal{B}_0^\alpha \rightarrow \mathcal{B}_0^\beta$ is weakly compact if and only if $I_g C_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}_0^\beta$ is compact if and only if

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^\beta |\varphi'(z)| |g(z)|}{(1 - |\varphi(z)|^2)^\alpha} = 0.$$

Remark 2. From the relationship of the composition operator C_φ and the integral-type operators J_g and I_g , we can obtain a complete characterization of the boundedness and compactness of the operators of C_φ , J_g and I_g . We omit the details.

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